Predictive Euler CFD - Resolution of NASA Vision 2030

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We show that computing turbulent solutions to Euler's equations with a slip boundary condition offers a Theory of Everything ToE for slightly viscous incompressible fluid flow as a parameter-free model, covering a vast area of applications in vehicle aero/hydrodynamics including airplanes, ships and cars. This work resolves the Grand Challenges of fluid dynamics described in NASA Vision 2030.

The foundation of the methodology is an extremely efficient Direct FEM Simulation (DFS) method. We describe a breakthrough in efficiency, allowing extremely small numerical dissipation by choosing very small stabilization coefficients, while allowing very large time step size.

This work is developed as part of the Digital Math framework [1] - as the foundation of modern science based on constructive digital mathematical computation. We invite you to run and modify the simulations yourself in your web browser. The Digital Math web environment with the Open Source Real Flight Simulator/FEniCS software for reproducing the results in the paper at in principle "zero" cost, together with more detailed presentation and results is available at:

http://digitalmath.tech/hiliftpw4-aiaa

We show that Euler CFD by the scientific method in Digital Math predicts drag, lift and pressure distribution in close correspondence with observations for real problems with complex geometry with specific focus on the 4th High Lift Prediction Workshop and so can serve to deliver complete realistic aero/hydro-data for simulators without input from model experiments in wind tunnel and towing tank or full-scale experiments, as a new revolutionary capability.

I. Euler CFD overview

The methodology is a Direct FEM Simulation (DFS) [2, 4] of the first principle Euler equations with a slip boundary condition - here denoted Euler CFD. The methodology is realized according to the scientific method in the Digital Math framework. We call this realization Real Flight Simulator (RFS).

These first principle equations are discretized by the Direct FEM approach, meaning Galerkin-Least-Squares (GLS) stabilization.

The Galerkin part of the method is formulated as below in FEniCS notation:

```
F_Galerkin = inner(udot + grad(u)*u + grad(p), v)*dx
F_Galerkin += inner(div(u), q)*dx
```

and in corresponding strong form in Latex notation:

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$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0 \text{ in } \Omega, \\ \nabla \cdot u = 0 \text{ in } \Omega, \\ u \cdot n = 0 \text{ on } \Gamma, \end{aligned} \tag{1}$$

An overview of the main ingredients of Euler CFD with the Digital Math RFS realization are given below, a more detailed description is in section VII:

Free slip boundary condition with 3D rotational slip separation

No thin boundary layers to resolve.

We show that the flow can separate with 3D rotational slip separation, at high velocity. See the 3D cylinder benchmark below for an illustration.

Our detailed validation of the reference benchmarks in the field: HiLiftPW2-4, NACA0012 wing, etc. all show that with only the pressure drag our results are within 5% of the experiment. This means skin friction drag is a small/negligible effect, which either can be omitted, or added as a minor adjustment.

In [5] we give an overview of both experimental and Euler CFD evidence, of low dependence of drag from Reynolds number in untripped configurations, consistent with free slip, from e.g. Abbott.

Automatic turbulence modeling by residual stabilisation

Through weighted least squares residual stabilisation generating *turbulent dissipation* $TD(t, u, p)$, as a solution to the open problem of *turbulence modeling* [29]. In particular, the weighted strong residual measures the *turbulent dissipation* as a mesh independent quantity meeting *Kolmogorov's K41 conjecture of finite turbulent dissipation* [73].

Adaptive adjoint-based a posteriori error control

Guaranteeing mesh-independence of drag and lift, and accuracy to a few percent in the validations.

Reproducibility

Guaranteeing the scientific method, allowing inspection, falsification, modification.

There is today a reproducibility crisis in science which we resolve with the Digital Math framework, and specifically here for Euler CFD with RFS.

II. Digital Math: Solving the reproducibility crisis

Run and modify the simulations yourself in your web browser!

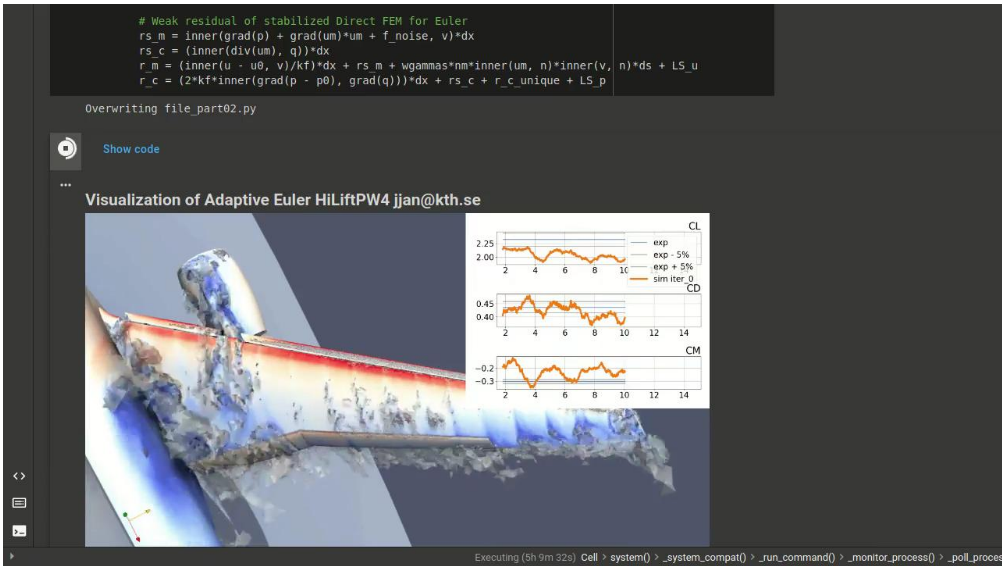
The Digital Math web environment with the Open Source Real Flight Simulator/FEniCS software for reproducing the results in the paper at in principle "zero" cost is available at:

http://digitalmath.tech/hiliftpw4-aiaa

together with more detailed results, articles, pedagogic material, etc.

Contact Johan Jansson (jjan@kth.se) for questions, comments and ideas.

Below is a screenshow of the Digital Math web environment with Euler Real Flight Simulator interactive quasi-real-time visualization:



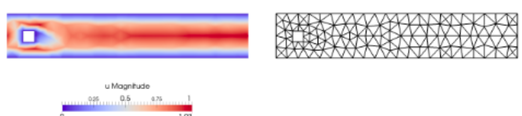
Automated Digital Math

We leverage our Open Source FEniCS framework, which automated the solution of partial differential equations by FEM, taking the mathematical notation as input. This allows an automation of Digital Math, described in a bird's eye view below:

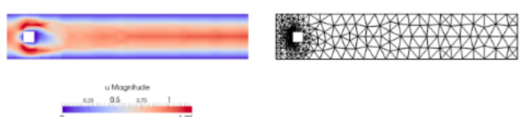
Automated discretization: (generate code for linear system from PDE/model.)

`r = (inner(grad(u), grad(v)) - inner(f, v))*dx` $\Rightarrow$ `Poisson.cpp`

Automated error control: (including parallel adaptive mesh refinement.)



$|M(e)| \leq TOL \Rightarrow$



with $M(e)$ a goal functional of the computational error $e = u - U$.

Goal: Automatically generate the **program**, **mesh** and **solution** from PDE/model (residual) and goal functional $M(U)$ (e.g. drag).

A. Extremely efficient stabilized Direct FEM

The predictive adaptive stabilized Direct FEM Simulation method takes the form:

```
F = F\-Galerkin + F\-Stab
```

where F_Stab is a residual-based Least Squares Galerkin stabilization of the form $(\delta R(U), R(v))$ with δ proportional to the mesh size h, controlled by the duality-based adaptive error control.

F_Stab provides numerical dissipation for the unresolved subscales in a Direct predictive and adaptive setting.

In previous work on DFS we have chosen $\delta = Ch$ or $\delta = Cmin(h)$ with $C \approx 1$. In this work we show a key breakthrough, choosing $C_U \approx 0.1$ for the velocity component U and $C_P \approx 0.01$ for the pressure component P. This corresponds to an efficiency increase of many orders of magnitude, which is key for the resolution of the Grand Challenges in fluid dynamics.

III. Validation

A. High Lift Prediction Workshop 4

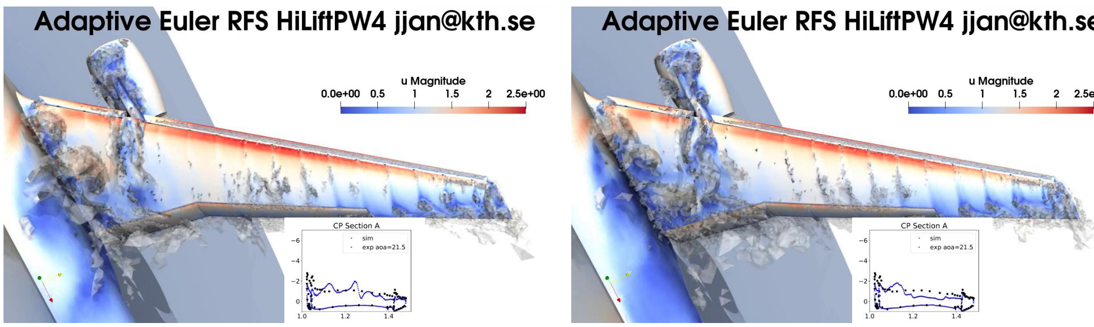


Fig. 1 Digital Math simulation of the High Lift Prediction Workshop 4 benchmark, a complete aircraft with slip BC, here showing stall at aoa=21.5 . The simulation predicts the lift and drag forces of the experiment through the range of angle of attack, including stall. Adaptive error control demonstrates mesh-independence in the figures below.

IV. Euler CFD as a solution to NASA Vision 2030

We see that Euler CFD with the Real Flight Simulator (RFS) realization already today in 2021 satisfies the goals of the NASA Vision 2030 challenges:

1. Emphasis on physics-based, predictive modeling

Euler/RFS is predictive by being first-principles, parameter-free and mesh/discretization independent by adjoint-based adaptive error control.

2. Management of errors and uncertainties resulting from all possible sources

Euler is first-principles, and does not have explicit modeling parameters. RFS relies on adjoint-based adaptive error control to guarantee mesh/discretization-independence, and additionally automatically generates the low-level source code from mathematical notation (just a few lines), thus eliminating the possibility of human bugs.

3. A much higher degree of automation in all steps of the analysis process

RFS relies on automated mesh generation based on adjoint-based adaptive error control, and additionally automatically generates the low-level source code from mathematical notation (just a few lines) including the adjoint formulation and the adjoint solution.

4. Ability to effectively utilize massively parallel, heterogeneous, and fault-tolerant HPC architectures

We demontrate that Euler/RFS has extremely cheap and fast performance (200 core hours), which allows an extreme effectiveness by being able to run a large number of simulations on in principle any parallel computer (also e.g. any virtual machine in a cloud setting, even in a web browser), at a cost affordable to any engineer, researcher or even student.

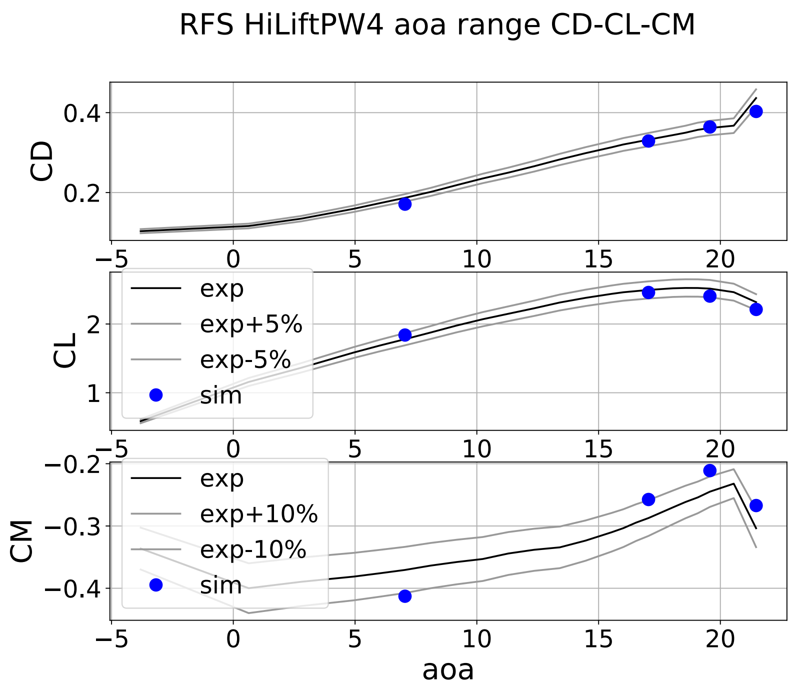


Fig. 2 Forces and pitch moment over a range of angles of attack. Prediction of CL and CD pre-stall within 5, and CM for all angles and CL and CD at stall within 10, specifically also predicts pitch-break.

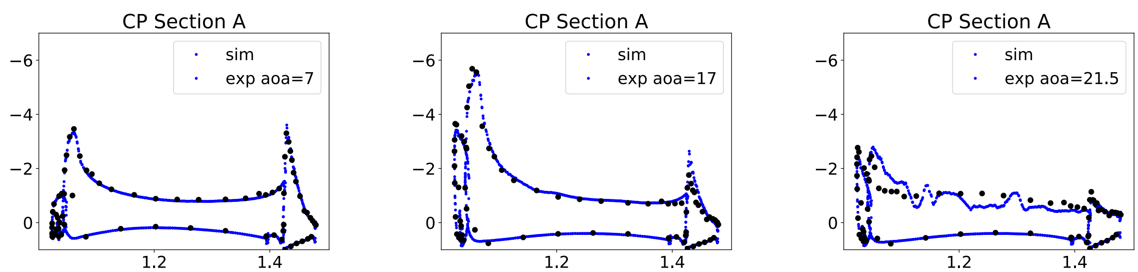


Fig. 3 CP plots at the A stattion for aoa=7, 17 and 21.5 showing excellent match to experiment.

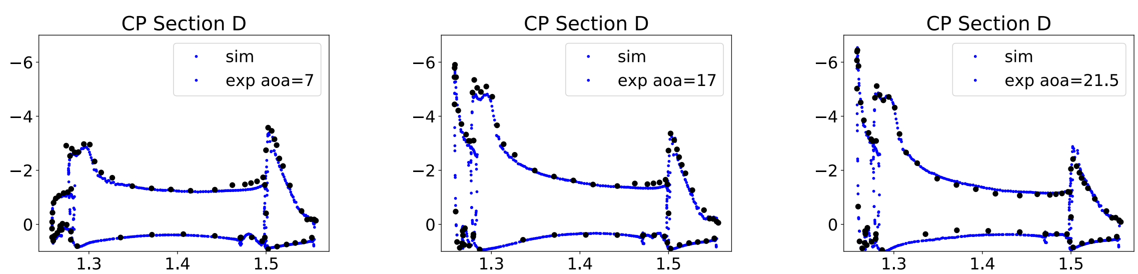


Fig. 4 CP plots at the D stattion for aoa=7, 17 and 21.5 showing excellent match to experiment.

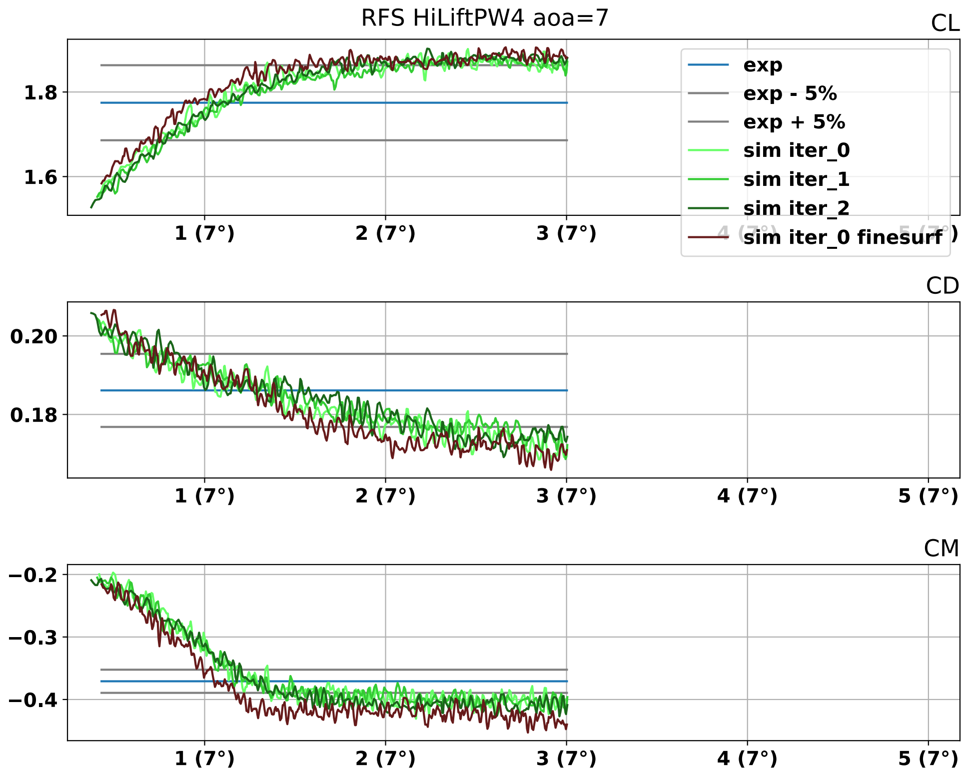


Fig. 5 Digital Math simulation of the High Lift Prediction Workshop 4 benchmark, a complete aircraft with slip BC. Adaptive error control demonstrates mesh-independence, here at aoa=7. Included is also a finer surface mesh, demonstrating independence of surface mesh resolution.

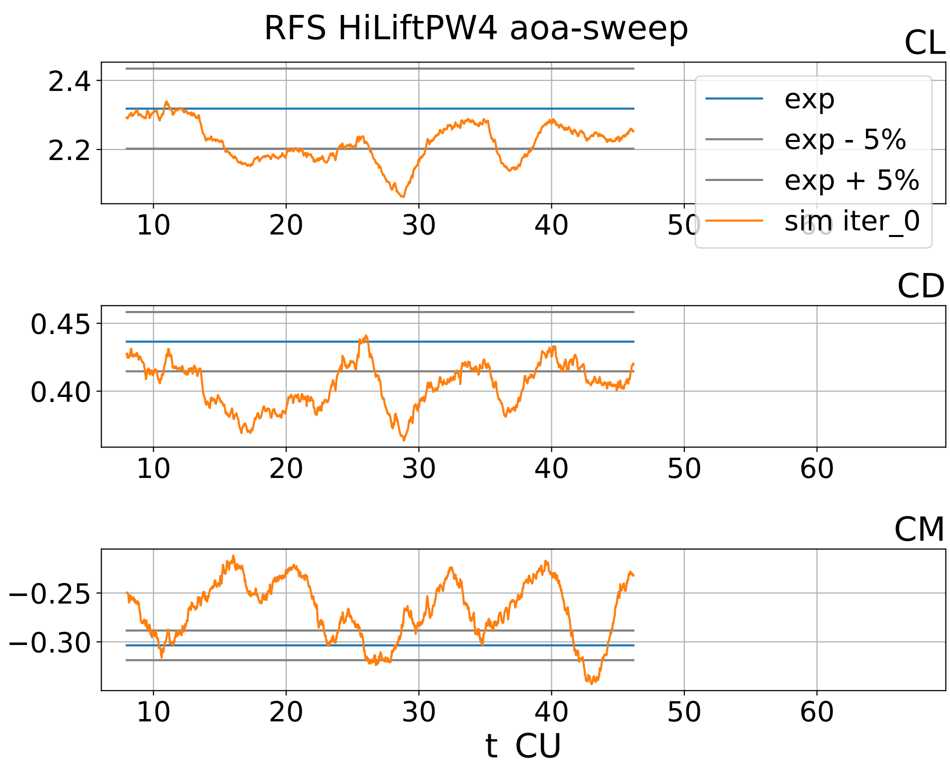


Fig. 6 Digital Math simulation of the High Lift Prediction Workshop 4 benchmark, a complete aircraft with slip BC. Longer time interval at stall, t aoa=21.5 .

5. Flexibility to tackle capability- and capacity-computing tasks in both industrial and research environments:

The same answer as above. We demontrate that Euler/RFS has extremely cheap and fast performance (200 core hours), which allows an extreme effectiveness by being able to run a large number of simulations on in principle any parallel computer (also e.g. any virtual machine in a cloud setting, even in a web browser), at a cost affordable to any engineer, researcher or even student.

6. Seamless integration with multidisciplinary analyses that will be the norm in 2030

Euler/RFS is realized in the Digital Math framework in FEniCS, taking the mathematical notation (just a few lines) as input and automatically generating the low-level source code. We have demonstrated general fluid-strucure interaction (FSI) generalizations in a very simple and automated way, and other multidisciplinary generalizations are possible or have been done in a similar way. Digital Math means an Open Source setting, where it's easy and natural to merge and integrate different formulations for e.g. different physical phenomena.

V. Euler's Dream

Read Euler, read Euler, he is the master of us all. (Laplace)

In 1755 the German mathematician Euler formulated a mathematical model describing the flow of air (subsonic) and water with the following prophetic declaration of *Euler's Dream* [24]:

- *My two equations contain all of the theory of fluid mechanics. It is not the principles of mechanics we lack to pursue this analysis but only Analysis (computation), which is not sufficiently developed for this purpose.*

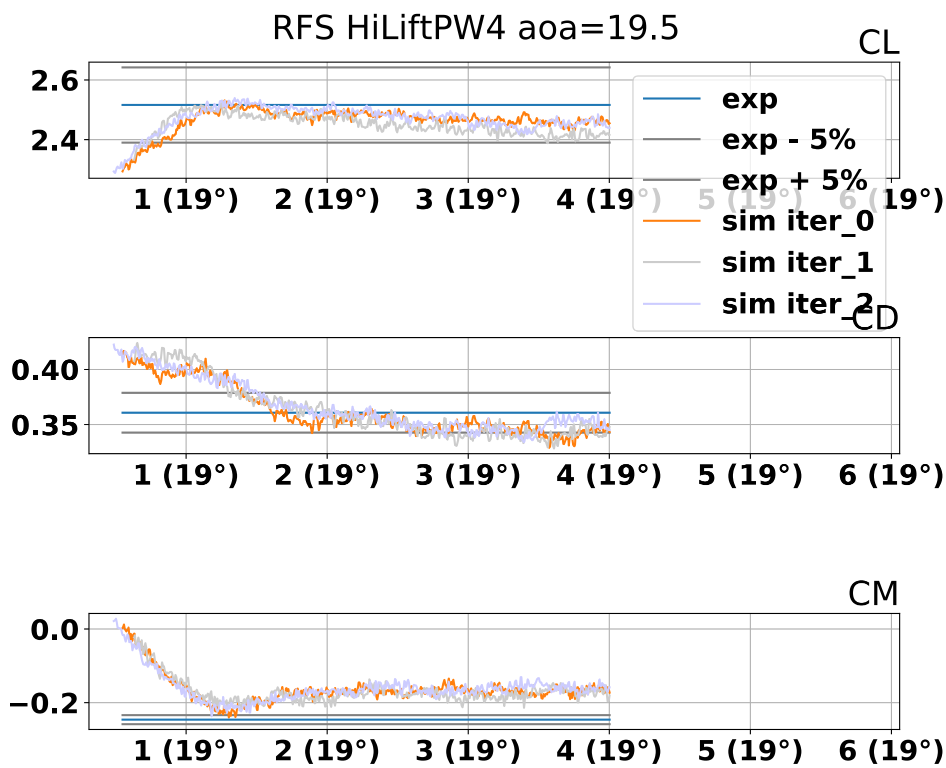


Fig. 7 Forces over a range of angle-of-attack, both CD and CL within 5% of the experiment and mesh-independent.

These are *Euler's equations* for (unit density) *slightly viscous incompressible* fluid flow formulated in terms of *fluid velocity* $u(x,t)$ and *fluid pressure* $p(x,t)$ depending on space-time coordinates (x,t) as an expression of *force balance*

(Newton's 2nd Law) and *incompressibility* complemented by a *slip boundary condition*. They read [17]:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= 0 \text{ in } \Omega, \\ \nabla \cdot u &= 0 \text{ in } \Omega, \\ u \cdot n &= 0 \text{ on } \Gamma, \end{aligned} \tag{2}$$

where Ω is a spatial domain occupied by the fluid with boundary Γ with unit normal n acting like a solid wall impenetrable to the fluid as expressed by $u \cdot n = 0$. The only forces acting on the fluid (without gravitation) are the internal pressure gradient ∇p as a volume force in Ω combined with a surface pressure force pn from the wall acting in the normal direction on Γ. Formally there are no internal viscous shear forces (*zero viscosity*) and no force tangential to the boundary (*zero skin friction*).

In *bluff body flow* Ω is the domain filled by fluid flowing past a volume occupied by a solid body at rest in a coordinate system with the flow velocity being constant at large distance from the body as a far-field condition. The basic problem in bluff body flow is to determine the pressure distribution from the fluid on the body with drag and lift as net forces opposite and perpendicular to the main flow direction in normalized form appearing as coefficients of drag C_D and and lift C_L. This is the basic problem of vehicle aero/hydrodynamics including airplanes, ships and cars. We shall see that computing *turbulent solutions* of Euler's equations allows accurate prediction of drag and lift for a body of arbirtary shape, as a realisation of Euler's Dream by computing.

As is clear from (2) the Euler equations are *parameter-free* since viscosity and skin friction parameters are set to zero. This means that the Euler equations/Euler's Dream represent Einstein's ideal mathematical model as a *Theory of Everything ToE* for a certain range of physics (slightly viscous incompressible) flow, that is a mathematical theory capable of making predictions about reality (drag and lift) without any input of parameters such as viscosity and skin friction. We give below massive evidence that computation of turbulent solutions to Euler's equations is a ToE for fluid mechanics, and as such very remarkable and useful. But it took 250 years to make computing powerful enough to make Euler's Dream come true, and the start for Euler in 1755 was rocky.

Eulers French adversary mathematician d'Alembert namely quickly crushed Euler's grand plan by showing that Euler's equations admitted certain solutions (potential solutions) showing zero drag and lift of a body moving through air or water, in direct contradiction to observation [23, 30, 33, 34]. This was coined *d'Alembert's Paradox* (in fact realised by Euler before 1755 [34]), which from start as expressed by Chemistry Nobel Laureate Hinshelwood, *separated practical fluid mechanics (hydraulics) describing phenomena (drag, lift), which cannot be explained, from theoretical fluid mechanics explaining phenomena (zero drag, lift), which cannot be observed.*

The paradox showed to resist all attempts of resolution by numbers of most able mathematicians, but zero lift is in particular incompatible with flight, and so had to be resolved at the dawn to modernity when powered human flight was shown to be possible by the Wright brothers in 1903. The young ambitious fluid mechanician Ludwig Prandtl took on the challenge by presenting a resolution in a 10 minute presentation at a 1904 mathematics conference at Heidelberg of a sketchy 8-page note *On flow motion with very small viscosity* [9, 27, 60]. Prandtl discriminated potential flow with zero skin friction claiming that a real fluid always meets a solid wall through a *boundary layer* with zero tangential relative velocity named *no-slip*, then apparently as a result of sufficient skin friction supposed to cause "flow separation by adverse pressure" and drag. Prandtl thus "resolved" d'Alembert's Paradox by declaring that the Euler equations with slip had to be expanded to the *Navier-Stokes equations* including boundary layer effects from positive viscosity with in particular a no-slip boundary condition as somehow an effect of viscosity, although admittedly "very small". But no-slip was an ad hoc assumption which Prandtl could not justify, since the exact nature of the microscopic or effective macroscopic contact between fluid and wall was unknown to him and so has remained into our days.

Prandt started out boldly declaring *I have now set myself the task to investigate systematically the laws of motion of a fluid whose viscosity is very small* [60] (same as Euler) with the plan of showing a big effect (substantial drag and lift) from a very small cause (very small viscosity), however as scientific problem something very delicate by asking for very detailed analysis. Large scale instability is a different thing.

Navier-Stokes equations as presented by Navier in 1823 (clarified by Saint-Venant 1930 [70] and Stokes [66] in 1842) were combined with a slip-friction boundary condition (shown below) and so Prandtl's no-slip was by no means a necessity, only a convenient assumption (to get rid of potential flow) as expressed by Prandtl [60]:

- *By far the most important question in the problem area is the behaviour of fluids at the walls of solid bodies. One does sufficient justice to the physical processes in the boundary layer between the fluid and the solid body if one assumes that the fluid adheres to the wall and that the velocity there is zero or correspondingly equal to the velocity of the body.*

Anyway, the fluid mechanics community was with the help of Prandtl relieved from a seemingly unresolvable most disturbing paradox as expressed by Hinshelwood, and so Prandtl in the 1920s was named *Father of Modern Fluid Mechanics* [62, 68] based on the Navier-Stokes equations with no-slip and not Euler's equations with slip.

But there was one main caveat: The Navier-Stokes equations with no-slip have solutions with boundary layers so thin that computational resolution is impossible with any forseeable computational power [53]. Prandtl's resolution thus came with the cost of making *Computational Fluid Dynamics CFD* into an impossibility asking for resolution of atomistic scales in a macroscopic setting, or complicated modeling.

In 2010, Hoffman and Johnson published [41] a different resolution of d'Alembert's Paradox showing that the reason zero-drag/lift of potential flow cannot be observed, is that potential flow as gradient of a potential satisfying Laplace's equation, is large scale unstable at separation as exact solution to Euler's equations, and so is replaced by solution with a turbulent wake after separation with drag/lift. In this article we now present Digital Math reproducible proof of this resolution, by computing *turbulent solutions* to Eulers equations with slip from a principle of best possible approximate solution with drag and lift in close agreement with observations, supported by mathematical analysis [39, 40]. Here large scale instability is not a small cause in the same sense as very small viscosity, and so is open to mathematical understanding. The new resolution was anticipated by Euler expecting separation being different from attachement [34, 57].

As a spin off a *New Theory of Flight* [37, 38] was developed revealing the true *Secret of Flight* [71] in physical terms, very different from the unphysical lifting line theory advocated by Prandtl as a follow up of the unhysical Kutta-Zhukowski circulation theory [16].

In 2017 Johan Jansson et. al. finally resolved the NASA Vision 2030 grand challenge by completing the methodology, and with Digital Math demonstrated prediction of stall in the Third High Lift Prediction Workshop [4]. Jansson showing that the critical addition of noise guaranteed the triggering of the instabilities in the 3D slip separation mechanism.

Stokes [66] suggested the possibility that a given flow motion does not imply its necessity [30] as expression of instability: *There may even be no steady mode of motion possible, in which case the fluid would continue perpetually eddying*. This was shown to be a reality in the new resolution 2010 of d'Alembert's Paradox. Stokes idea was expressed by the mathematician Garrett Birkhoff in the 1st edition 1950 of his *Hydrodynamics* [15], but was removed in the 2nd edition because of harsh critcism from the fluid dynamics community and so only managed to resurface 55 years after Prandtl's death in the new resolution.

Darrigol [30] (also [7]) gives a detailed exposition of early work on hydrodynamic stability by Stokes, Helmholtz, Lord Kelvin and Reynolds, as well as attempts to resolve d'Alembert's paradox by Rayleigh, Poncelet and Saint-Venant followed by that by Father Prandtl, which became the resolution serving the modern fluid mechanics of the 20th century, although: *In summary, Prandtl's early insights into boundary-layer theory did not bring him much closer to a practical solution of low-viscosity resistance problems. The difficulties of the determination of separated flow remain unsolved to this day*.

We now present massive evidence collected in this article in Digital Math reproducible form showing that computing turbulent solutions of Euler's equations with slip, which we will refer to as *Euler CFD*, opens basically all of slightly viscous nearly incompressible flow to predictive simulation without parameter input and need to resolve thin no-slip boundary layers, thus with readily available computing power, all along Euler's visionary prophecy.

Euler thus was right in predicting that his dream would come true once the computing power (Analysis) was strong enough to compute turbulent solutions of the Euler equations with slip, and Prandtl was wrong claiming drag and lift to be effects of unresolvable thin no-slip boundary layers making CFD into an impossibility. It took 250 years, but now it is here and sets a new standard in engineering by computational mathematics opening a window to the Clay Millennium Problem on Navier-Stokes equations [18] presented as follows:*Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations*. It seems that the secrets were hidden already in the Euler equations and can now be revealed by Euler CFD.

VI. Digital Math: Scientific Automated Flow Simulation

The scientific process has not kept up with digital technology, and there is today a reproducibility crisis. Lorena Barba representing NASEM describes the situation as:

"The widespread use of computation and large volumes of data have transformed most disciplines of science and enabled new and important discoveries. But this revolution is not yet reflected in the ways that scientific findings are published and shared with the relevant communities. Extending the scholarly record to data, software, and computational environments and workflows is a must to ensure the robustness of science in this digital era."

We present the Digital Math framework as the foundation for modern science based on constructive digital mathematical computation, and as a solution to the reproducbility crisis. The computed result (coefficient vector, FEM function, plot, etc.) is a mathematical theorem, and the mathematical Open Source code, here in the FEniCS framework, and computation is the mathematical proof. We can also derive additional constructive proofs from the FEniCS and FEM formulation, such as stability.

Digital Math represents digitalization of science, mathematics, society and industry in the form of automated and easily understandable computation of mathematical models. It is here realized in the Open Source FEniCS framework with world-leading performance and recognized at the highest level of science and industry together with an effective pedagogical concept with combined abstract theory and mathematical interactive programming in a "one-click" cloud-HPC web-interface, accessible to anyone: In the Digital Math HiLiftPW4 site (http://digitalmath.tech/hiliftpw4-aiaa), the full adaptive Euler CFD methodology for several cases is available, from a cube benchmark case [3] to aircraft for you to inspect, run, modify, and reproduce, just as all examples presented. The web environment is illustrated in Figure 8.

Fig. 8 Digital Math Euler CFD simulation, visualization, editing in mathematical notation, of an electric aircraft connected to the ELISE project for electric aviation.

Computational solution of turbulent solutions of Euler's equations as *Euler CFD* is automated using FEniCS [31] for automation of the finite element discretisation used to express the principle of best possible approximate solution. This brings a new tool of *Automated Flow Simulation* with only geometry input, which in particular allows for the first time computation of complete aero-data (forces) for any given airplane/car/ship for design, Digital Twins, interactive simulators, etc. Euler CFD can include (small) positive boundary friction allowing also flow before drag crisis to be computed, but it introduces friction as a coefficient to be fittted to experiments. For high Reynolds number beyond drag crisis - the regime relevant to aerodynamics - this is not needed, and Euler CFD is completely parameter-free.

VII. Euler CFD: Turbulent Solutions of Euler's Equations

Turbulent solutions to Euler's equations as Euler CFD are computed as *best possible approximate solutions* in the sense of having residuals which are small in a weak sense and not too large in a strong and sense, in a situation when all solutions with small strong residuals (laminar solutions and potential solutions in particular) are unstable (as solutions to Euler's equations) and thus do not persist over time. We here face a new situation where *only turbulent flow is computable* and laminar not, as an expression of the fluctuating nature of turbulence as a consequence of local exponential instability, as seen in a waving flag showing the only motion which can persist.

More precisely, the best possible aspect is realised by a finite element method augmenting small weak residuals from

Galerkin orthogonality with weighted least squares control of strong residuals, the latter introducing a viscous effect as a form of *turbulent viscosity* set by computation alone without need to model or measure turbulent viscosity beyond human comprehension. It connects to Leibniz idea of the Real World as a *Best Possible World*; the flag is doing its best possible as well as Euler CFD showing mesh-independence of drag and lift with readily available computing power.

Euler CFD takes the following space-discrete variational form: Find $(u(t), p(t)) \in V \times Q$ such that for $t > 0$, for all $(v, q) \in V \times Q$

$$\begin{aligned} (\tfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p, v)_\Omega + (\alpha_1 h u \cdot \nabla u, u \cdot \nabla v)_\Omega = 0, \\ (\alpha_2 h^{\frac{3}{2}} \nabla p, \nabla q)_\Omega = -(\nabla \cdot u, q)_\Omega, \end{aligned} \tag{3}$$

where $(\cdot,\cdot)_\Omega$ denote $L_2(\Omega)$ inner products, V and Q are finite element spaces of continuous piecewise linear velocity-pressure functions (v, q) satisfying $v \cdot n = 0$ and $\frac{\partial q}{\partial n} = 0$ on Γ and appropriate far-field conditions, $h = h(x)$ local finite element size, and $\alpha_1 \approx 0.1$ and $\alpha_2 \approx 0.1$ are constants chosen from a principle of best possible with respect to output measured by the solution to an associated linearised dual problem. Time stepping is performed using continuous piecewise linear trial functions and piecewise constant test functions in time (Crank-Nicolson type). Euler CFD shows insensitivity within quite wide margins to the precise choice of α_1 and α_2, as well as mesh size once small enough to resolve geometry and separation.

Euler CFD performs *automatic turbulence modeling* through weighted least squares residual stabilisation of the momentum equation combined with pressure stabilisation with corresponding *turbulent dissipation* $TD(t, u, p)$ obtained choosing $(v, q) = (u(t), p(t))$ in (3):

$$TD(t, u, p) = (\alpha_1 h u(t) \cdot \nabla u(t), u(t) \cdot \nabla u(t))_\Omega + (\alpha_2 h^{\frac{3}{2}} \nabla p, \nabla p)_\Omega \tag{4}$$

with $u \cdot \nabla u$ as a representative part of the full momentum residual $R(u) = \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p$ when not small. Turbulence is then identified by substantial turbulent dissipation $TD(t, u, p)$ over time (showing mesh size independence) as an expression of impossbility of computing a solution over time with small momentum residual in a strong ($L_2(\Omega)$) sense and $|\nabla u| \sim h^{-\frac{1}{2}}$ in regions of turbulence.

Residual stabilisation of the momentum equation is necessary because weak satisfaction in the form $(\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p, v)_\Omega = 0$ for all smooth v, does not itself give control of the *kinetic energy* $\frac{1}{2}(u(t), u(t))_\Omega$, the reason being that it is not feasible to choose $v = u(t)$ because $u(t)$ is not smooth.

Euler CFD includes *adaptive duality based a posteriori error control* guaranteeing drag and lift up to a few percent. A key element is here the oscillating character of turbulent flow leading to *cancellation in the linearised dual problem* making the *dual solution computable* and properly bounded [39].

Euler CFD includes *automatic turbulence modeling* through *weighted strong residual control* as a dissipative effect with a complex flow dependence beyond viscous shear stress. It appears as a solution to the open problem of *turbulence modeling* [29]. In particular, the weighted strong residual measures the *turbulent dissipation* as a mesh independent quantity meeting *Kolmogorov's K41 conjecture of finite turbulent dissipation* [73].

VIII. Predictive Euler CFD - Resolution of the paradox

We show that predictive Euler CFD resolves the paradox. The potential solution with zero drag is unstable - shown by stability analysis and computational evidence with adaptive error control. We illustrate the resolution by the basic cylinder model problem, and also by the most advanced benchmark in the world representing vehicles and aerodynamic devices - the High Lift Prediction Workshop, where we show that Euler CFD predicts the experiment to 5% with mesh independence, and predicts they key stall mechanism.

In Figure 9 we show our resolution of the paradox with Digital Math Euler CFD: the potential solution is unstable and develops streamwise vortices on the downstream side of the cylinder, generating "3D slip separation" - separation at high flow velocity.

In the HiLiftPW4 results throughout this article, we show prediction of stall, both CD and CL within 5% of the experiment and mesh-independent. This represents the resolution of the NASA Vision 2030 grand challenge, and with Digital Math guaranteeing the scientific method.

IX. Conclusions

We have showed that computing turbulent solutions to Euler's equations with a slip boundary condition offers a Theory of Everything ToE for slightly viscous incompressible fluid flow as a parameter-free model, we are now able to

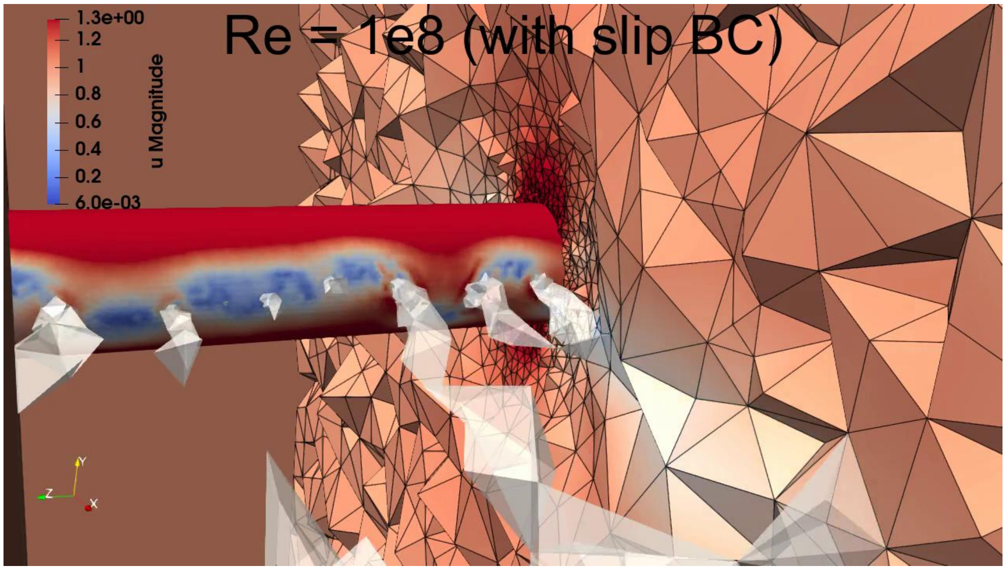


Fig. 9 Digital Math simulation of a 3D cylinder with slip BC, and a sweep over the Reynolds number/viscosity. Starting at Re=1e9 (low viscosity), we observe the "3D slip separation" with streamwise vortices generated on the downstream side of the cylinder. However, by Re=1e3 and Re=1e4 this mechanism is damped out.

predict a vast area of applications in vehicle aero/hydrodynamics including airplanes, ships and cars. This work resolves the Grand Challenges of fluid dynamics described in NASA Vision 2030.

Key specific results are breakthrough results validating RFS for the High Lift Prediction Workshops and the NACA0012 aoa=0 case, demonstrating that RFS predicts the regime of cruising aircraft and ships (hydrodynamics). Additionally we show Digital Math simulations of similar applications such as the full car DrivAer automotive standard benchmark, etc.

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